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Rabi oscillations in a two-level atomic system with a pseudo-Hermitian Hamiltonian

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Abstract

Rabi oscillations of a two-level atomic system with damping effects are described by the use of a pseudo-Hermitian Hamiltonian. The Rabi frequency and the amplitudes for being in the upper and lower level depend in the present system on both the Hermitian resonant interaction between the em field and the atomic dipole and on damping effects. The time development of the two-level system is studied by using the metric of the pseudo-Hermitian Hamiltonian and its bi-orthonormal basis of states. The special characteristics of the present system are related to PT (parity-time reversal) and C (generalized conjugation operator) invariant properties of the pseudo-Hermitian Hamiltonian.

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1. Introduction

A basic postulate of quantum mechanics is that observables are represented by Hermitian Hilbert space operators and that their time propagators are given by unitary operators. It has been shown, however, that non-Hermitian operators and Hamiltonians are very efficient for treating scattering phenomena in atomic and molecular physics [1, 2]. Bender *et al* [3–7] have suggested replacing the condition of self-adjointness by the weaker condition of PT (parity-time reversal) symmetry, which leads sometimes to real eigenvalues of the non-Hermitian Hamiltonian. By using an additional C (conjugation operator) symmetry [5], an inner product whose associated norm is positive definite could be constructed. The concept of a pseudo-Hermitian property has been introduced [8–12] and it has been shown that the interesting spectral properties of the PT-symmetric Hamiltonian follow from their pseudo-Hermiticity. The concept of pseudo-Hermiticity is developed by following a definition of 'distorted' inner product: $\langle\psi|\eta\psi\rangle$ where η is called the metric. A Hamiltonian is called pseudo-Hermitian if [8–12]

$$\eta H \eta^{-1} = H^{\dagger}. \tag{1}$$

Pseudo-Hermiticity is a general condition on a Hamiltonian for possessing real eigenvalues, and its relation with PT symmetry is important in relation to physical systems. Various mathematical properties of pseudo-Hermitian Hamiltonians have been analysed [3–13]. There is an enormous amount of published literature on pseudo-Hermitian Hamiltonian systems. We have referred here to a few articles on which the present analysis is based (see additional references in [3–13]).

In this paper we use a pseudo-Hermitian Hamiltonian to treat a certain problem taken from the field of quantum optics. The analysis leads to physical results which may be checked experimentally.

We assume an atomic system where the atom is excited into a superposition of two levels which are above the ground state. The wavefunction of the two-level atomic system is given by

$$|\psi\rangle = C'_a|\phi'_a\rangle + C'_b|\phi'_b\rangle, \tag{2}$$

where a and b denote, respectively, the upper and lower level of these two excited levels. C'_a and C'_b are the amplitudes of being in these two levels where the amplitudes are functions of time due to the interactions:

(a) resonant interaction between monochromatic em field and the atomic dipole moment between the two levels.
(b) Damping of the atoms from the upper and lower level is described by phenomenological decay constants γ_a and γ_b, respectively.

Under these assumptions the amplitudes of the two levels are developed in time, in the interaction picture and in the rotating wave approximation on resonance as [14]

$$\begin{aligned} \mathrm{i}\dot{C}'_a &= -\mathrm{i}\gamma_a C'_a + V C'_b, \\ \mathrm{i}\dot{C}'_b &= -\mathrm{i}\gamma_b C'_b + V^* C'_a \quad (\hbar = 1) \end{aligned} \tag{3}$$

where V is the radiation–atom interaction matrix element between the two levels.

Defining

$$C_a = C'_a \exp[1/2(\gamma_a + \gamma_b)t], \qquad C_b = C'_b \exp[1/2(\gamma_a + \gamma_b)t], \tag{4}$$

we get after a straightforward algebra:

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} C_a \\ C_b \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\frac{(\gamma_a-\gamma_b)}{2} & V \\ V^* & \mathrm{i}\frac{(\gamma_a-\gamma_b)}{2} \end{pmatrix}\begin{pmatrix} C_a \\ C_b \end{pmatrix}, \tag{5}$$

which can be written as

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} C_a \\ C_b \end{pmatrix} = H\begin{pmatrix} C_a \\ C_b \end{pmatrix}; \qquad H = \begin{pmatrix} -\mathrm{i}\gamma & V \\ V^* & \mathrm{i}\gamma \end{pmatrix}; \qquad \gamma = \frac{(\gamma_a - \gamma_b)}{2} \quad (\hbar = 1). \tag{6}$$

The Hamiltonian of equation (6) is pseudo-Hermitian as it obeys the conditions of theorem 3 of [13] 'a traceless 2×2 matrix is pseudo-Hermitian if and only if it has a real determinant'. This pseudo-Hermitian Hamiltonian has certain properties and symmetries which will be obtained here by the methods given in [3–13]. One should take into account that, although all the following analysis concentrates on solutions of the Schrödinger equation for Hamiltonian (6), the amplitudes C'_a and C'_b are obtained from C_a and C_b by multiplying the latter by the phenomenological common exponential decay constant $\exp[-(1/2)(\gamma_a + \gamma_b)t]$. In principle one can consider the possibility of designing a two-level system which will follow the use of equation (6) directly, but the use of the present transformation from equations (3) to (6) can be more easily exploited experimentally. One should take into account that, for small

deviations from resonance, equations (3) should be modified [14] but for simplicity we limit the treatment to exact resonance.

This paper is arranged as follows: in section 2 we relate the solutions of (6) to its properties as a pseudo-Hermitian Hamiltonian with real eigenvalues. In section 3 we discuss the symmetries of this pseudo-Hermitian Hamiltonian. In section 4 we summarize our results.

2. The time development of the two-level system under the pseudo-Hermitian Hamiltonian with real eigenvalues

The Hamiltonian (6) is diagonalizable for any combination of γ and V with the following values for the eigenvalues:

$$E_{\pm} = \pm\sqrt{|V|^2 - \gamma^2}. \tag{7}$$

For simplicity of notation we will denote E_+ as E and E_- as $-E$. The similarity matrix associated with the diagonalization is

$$D = \frac{1}{E}\begin{pmatrix} \sqrt{|V|^2-\gamma^2} - \mathrm{i}\gamma & -\sqrt{|V|^2-\gamma^2} - \mathrm{i}\gamma \\ V^* & V^* \end{pmatrix} = \frac{1}{E}\begin{pmatrix} E - \mathrm{i}\gamma & -E - \mathrm{i}\gamma \\ V^* & V^* \end{pmatrix} \tag{8}$$

and a set of eigenvectors is given by

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{E}\begin{pmatrix} \sqrt{|V|^2-\gamma^2} - \mathrm{i}\gamma \\ V^* \end{pmatrix} = \frac{1}{E}\begin{pmatrix} E - \mathrm{i}\gamma \\ V^* \end{pmatrix} \\ |\psi_2\rangle &= \frac{1}{E}\begin{pmatrix} -\sqrt{|V|^2-\gamma^2} - \mathrm{i}\gamma \\ V^* \end{pmatrix} = \frac{1}{E}\begin{pmatrix} -E - \mathrm{i}\gamma \\ V^* \end{pmatrix}. \end{aligned} \tag{9}$$

We find that under the condition $|V|^2 > \gamma^2$ the energies are real although the Hamiltonian is non-Hermitian. This condition is especially interesting in quantum optics as we get Rabi oscillations which are different from the ordinary ones. We analyse our system under this condition (we will comment at the end of this section on the solutions under the conditions $|V|^2 < \gamma^2$, and $|V|^2 = \gamma^2$). We use the following theorem proved in [9] which helps us in fixing a certain definite metric:

> If a pseudo-Hermitian $n \times n$ matrix H admits real eigenvalues $(E_1, E_2 \ldots)$ and D is its diagonalizing matrix, then H is η pseudo-Hermitian where $\eta = (DD^\dagger)^{-1}$. The converse is also true.

One should take into account that it is possible to choose another diagonalizing matrix D and consequently another choice of eigenvectors and metric operator η_+. The above theorem does not eliminate the non-uniqueness of η_+ but uses a specific set of eigenvectors and correspondingly a specific metric η_+ by which we make the present analysis. Using this theorem we construct the positive definite metric

$$\eta_+ = (DD^\dagger)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \frac{\mathrm{i}V\gamma}{|V|^2} \\ -\frac{\mathrm{i}V^*\gamma}{|V|^2} & 1 \end{pmatrix}. \tag{10}$$

As a consequence of the metric η_+ we get the positive definite inner product:

$$\langle\psi_m|\eta_+\psi_n\rangle = \langle\psi_m|\Gamma_n\rangle = \delta_{mn} \tag{11}$$

where $\langle\psi_m|$ and

$$|\eta_+\psi_n\rangle = |\Gamma_n\rangle \tag{12}$$

constitute the bi-orthonormal basis. Using (9), (10) and (12) we get

$$|\Gamma_1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}\frac{\mathrm{i}\gamma}{V} + \frac{E}{2V} \end{pmatrix}; \qquad |\Gamma_2\rangle = \begin{pmatrix} -\frac{1}{2} \\ \frac{\mathrm{i}\gamma}{2V} + \frac{E}{2V} \end{pmatrix}. \tag{13}$$

One can easily check that $\langle\psi_m|$ $(m = 1, 2)$ of (9) and Γ_n $(n = 1, 2)$ of (13) obey the bi-orthonormality conditions (11).

We can use the above formalism for evaluating the time development of the two-level system under the pseudo-Hermitian Hamiltonian (6) with the real eigenvalues $E_+ = -E_- = E$. The initial two-level wavefunction can be described as a superposition of the eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$ of the pseudo-Hermitian Hamiltonian:

$$|\psi(0)\rangle = C_1(0)|\psi_1\rangle + C_2(0)|\psi_2\rangle, \tag{14}$$

where $C_1(0)$ and $C_2(0)$ are the amplitudes of $|\psi_1\rangle$ and $|\psi_2\rangle$, correspondingly, at time $t = 0$.

The time development of this state is given as

$$|\psi(t) = C_1(t)|\psi_1\rangle + C_2(t)|\psi_2\rangle \tag{15}$$

where

$$C_1(t) = C_1(0)\exp(-\mathrm{i}Et); \qquad C_2(t) = C_2(0)\exp(\mathrm{i}Et). \tag{16}$$

We find that a simple time development of the state is obtained by representing it as a linear combination of the eigenstates $|\psi_1\rangle$ and $|\psi_2\rangle$, as given in (15) and (16). Physically it is convenient to represent the state $|\psi\rangle$ by the amplitudes $C_a(t)$ and $C_b(t)$ of being in the upper and lower level, respectively, as given by (6). The transformations between these two bases of the quantum two-level state are given by

$$\begin{pmatrix} C_a(t) \\ C_b(t) \end{pmatrix} = \frac{1}{E}\begin{pmatrix} E - \mathrm{i}\gamma & -E - \mathrm{i}\gamma \\ V^* & V^* \end{pmatrix}\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = D\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} \tag{17}$$

and the inverse transformation is given by

$$\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{E+\mathrm{i}\gamma}{V^*} \\ -1 & \frac{E-\mathrm{i}\gamma}{V^*} \end{pmatrix}\begin{pmatrix} C_a(t) \\ C_b(t) \end{pmatrix} = D^{-1}\begin{pmatrix} C_a(t) \\ C_b(t) \end{pmatrix}. \tag{18}$$

We find according to equation (17) that

$$C_a(t) = C_1(t) - C_2(t) - \frac{\mathrm{i}\gamma}{E}(C_1(t) + C_2(t)), \tag{19}$$

$$C_b(t) = \frac{V^*}{E}(C_1(t) + C_2(t)). \tag{20}$$

We demonstrate the calculation of the time development of the two-level system under the pseudo-Hermitian Hamiltonian with real eigenvalues, in an example, assuming that the atom is initially in the upper level, i.e., assuming

$$C_a(0) = 1; \qquad C_b(0) = 0. \tag{21}$$

Then, according to (18), for this example

$$C_1(0) = \tfrac{1}{2}; \qquad C_2(0) = -\tfrac{1}{2}, \tag{22}$$

and according to (16):

$$C_1(t) = \tfrac{1}{2}\,\mathrm{e}^{-\mathrm{i}Et}; \qquad C_2(t) = -\tfrac{1}{2}\,\mathrm{e}^{\mathrm{i}Et}. \tag{23}$$

Using equations (19) and (20) we get for this example:

$$C_a(t) = \cos(Et) - \frac{\gamma}{E}\sin(Et), \tag{24}$$

$$C_b(t) = -\mathrm{i}\frac{V^*}{E}\sin(Et). \tag{25}$$

It is easy to check that $C_a(t)$ and $C_b(t)$ of equations (24) and (25) with the initial conditions $C_a(0) = 1, C_b(0) = 0$ obey equation (6). One finds that the normalization condition $|C_a|^2 + |C_b|^2 = 1$ is not obeyed here since the pseudo-Hamiltonian (6) includes in addition to the Hermitian Rabi matrix element also damping and amplifying terms (related to $\pm\mathrm{i}\gamma$ in the Hamiltonian). Although we have demonstrated here explicit calculation of $C_a(t)$ and $C_b(t)$ for certain initial conditions, the present procedure of calculations can be used for any initial conditions.

We can summarize the following properties for the time development of the present system:

(a) Although the present two-level system includes two decay constants γ_a and γ_b (see (3)), for cases for which $|V|^2 > \gamma^2$ the whole system decays only by the average decay constant $\frac{\gamma_a+\gamma_b}{2}$.
(b) By transforming the equations of motion to this decaying system (see (4)), we get equations of motion described by a pseudo-Hermitian Hamiltonian (see (6)).
(c) The 'pseudo-Hermitian Rabi oscillations' are in frequency $\sqrt{|V|^2 - \gamma^2}$ which is different from the Rabi frequency $|V|$ for the usual two-level Hermitian Hamiltonian on resonance.
(d) The amplitudes $C_a(t)$ and $C_b(t)$ for the atom to be in the upper and lower state, respectively, depend on both the Rabi matrix element and the damping and amplifying parameters $\pm\gamma$.
(e) The present pseudo-Hermitian Hamiltonian is described in the frame in which the whole atom is decaying by the average decaying constant $\frac{\gamma_a+\gamma_b}{2}$ and the transformation of the amplitudes to the ordinary frame (i.e., to $C'_a(t)$ and $C'_b(t)$) can be made by using (4).
(f) The eigenvectors of the pseudo-Hermitian Hamiltonian obey the bi-orthonormality condition based on the metric η_+ (see (10) and (11)).
(g) Summarizing the present analysis, we find that the properties of the present system are related to solutions of the Schrödinger equation for Hamiltonian (6). Also we find that the present pseudo-Hermitian Hamiltonian includes certain symmetries which are discussed in the following section.

Finally, let us mention briefly what happens if $|V|^2 - \gamma^2 < 0$. In this case the eigenvalues of H are purely imaginary. However, H still fulfils the conditions of theorem 3, [13], and hence it is pseudo-Hermitian. Therefore, equations (17)–(23) remain valid. The only difference is in equations (24) and (25). These become

$$C_a(t) = \cosh(\sqrt{\gamma^2 - |V|^2}t) - \frac{\gamma}{\sqrt{\gamma^2 - |V|^2}}\sinh(\sqrt{\gamma^2 - |V|^2}t) \tag{26}$$

$$C_b(t) = \frac{-\mathrm{i}V^*}{\sqrt{\gamma^2 - |V|^2}}\sinh(\sqrt{\gamma^2 - |V|^2}t). \tag{27}$$

Hence we have two different exponential decay modes

$$\exp\left\{-\frac{(\gamma_a + \gamma_b)}{2} \pm \sqrt{\gamma^2 - |V|^2}\right\}t$$

without any oscillations. Also, the critical situation $|V|^2 - \gamma^2 = 0$ can be easily solved to yield

$$C_a(t) = 1 - \gamma t; \qquad C_b(t) = -\mathrm{i}V^* t. \tag{28}$$

(Note that in this case $|V| = \gamma$.)

Since these are multiplied by $\exp\left[-\left(\frac{\gamma_a+\gamma_b}{2}\right)t\right]$, we have an initial linear dependence followed by basically exponential decay. Since our solutions (24)–(28) should be multiplied by the exponential decay $\exp\left[-\left(\frac{\gamma_a+\gamma_b}{2}\right)t\right]$ any divergences [15] do not occur in our system.

3. C-, PT- and CPT-invariance of the present pseudo-Hermitian Hamiltonian

Usually the 'time reversal' and the 'spatial inversion' transformations are defined as $(t, \vec{x}) \rightarrow (-t, \vec{x})$ and $(t, \vec{x}) \rightarrow (t, -\vec{x})$, respectively. The latter transformation changes right-handed spatial coordinate system to left-handed coordinate system and it is, therefore, referred to as a 'parity' transformation.

We follow here the method introduced by Bender *et al* [5], for describing the symmetries of our pseudo-Hermitian Hamiltonian. They defined the P operator of the two-level system as

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{29}$$

so that $P = P^{-1}$, $P^2 = 1$.

There are certain degrees of freedom in choosing the C-, PT- and CPT-operators. For example, Mostafazadeh [11] and Ahmed [9] have defined 'generalized P operator' which for our system is constructed as

$$P = \sum_{n=1}^{2} (-1)^n \psi_n \psi_n^\dagger = \begin{pmatrix} 0 & -\frac{2V}{E} \\ -\frac{2V^*}{E} & 0 \end{pmatrix}. \tag{30}$$

We find that in definition (30) the operator P is proportional to the dipole interaction matrix element. The common property of definitions (29) and (30) is that the operator P introduces transitions between the two states which have an opposite parity (related to the property of the dipole moment).

The non-uniqueness of the P and T operators follows from the non-uniqueness of the choice of eigenvectors and metric operator η_+ as explained previously concerning the use of equations (8) and (9). The antilinear operator T is described in the general case by the multiplication of a unitary operator U by K_0 where K_0 is a complex conjugation [16]:

$$T = U K_0. \tag{31}$$

Note that $K_0^2 = 1$ and therefore $T^{-1} = K_0 U^\dagger$.

Following Bender *et al* [5] we define for our system $U = 1$ so that T becomes equivalent to K_0. Different definitions for the time reversal operator T have been introduced by Mostafazadeh [11] and Ahmed [9], but we use the simple representation [5]:

$$T = K_0. \tag{32}$$

We find that PT commutes with the Hamiltonian as

$$P K_0 H K_0^{-1} P^{-1} = H. \tag{33}$$

In comparison to the ordinary Hermitian Hamiltonian which obeys the relation $H = H^\dagger$, we find that the present pseudo-Hermitian Hamiltonian of equation (6) obeys the relation

$$H^2/E^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{34}$$

This equality can be related to another symmetry of the present pseudo-Hermitian Hamiltonian. Using the generalized definition of conjugation operator [11] we get for the present system

$$C = \sum_{n=1}^{2} (-1)^n \psi_n \Gamma_n^\dagger = \frac{1}{E} \begin{pmatrix} \mathrm{i}\gamma & -V \\ -V^* & -\mathrm{i}\gamma \end{pmatrix} = -H/E. \tag{35}$$

Weigert [17] has shown that the conjugation operator is a function of the Hamiltonian. This fact is verified for the present system where C is a simple function of the Hamiltonian. We find in agreement with (35) that $C^2 = 1$, and that C commutes with the Hamiltonian in a quite trivial way. This invariance property eliminates negative inner products [5] as expressed by the metric η_+ of (10) and the positive inner products of (11). For more general properties of pseudo-Hermitian Hamiltonians we refer to the literature [3–13, 15, 17–19].

4. Discussion and conclusions

In this work we have analysed a certain problem taken from the field of quantum optics by exploiting the methods introduced in the literature [3–19] for treating pseudo-Hermitian Hamiltonians. We have treated the time development of a two level atomic system in which a resonant classical em field leads to dipole interaction between the two levels, and damping of the atoms from these two levels is described by phenomenological damping constants. For cases for which the dipole interaction is large relative to the damping effects (i.e., above a certain critical value) the whole two level system decays only by the averaged decay constant and the probability to be in the two decaying states shows oscillations between these two levels. Although these oscillations have a certain similarity to Rabi oscillations which is a very fundamental effect in quantum optics [1, 14], the present analysis by the use of a certain pseudo-Hermitian Hamiltonian shows new interesting effects. The 'pseudo Rabi frequency' and the amplitudes of being in the two levels depend in addition to the dipole interaction matrix element also on the damping and amplifying processes included in the pseudo-Hermitian Hamiltonian. In developing the present pseudo-Hermitian Hamiltonian we have used definitions of inner products related to a certain metric and introduced bi-orthogonal basis of states depending on this metric. The relations between the present pseudo-Hermitian Hamiltonian and certain symmetries known as P-, PT- and CPT-symmetries have been evaluated. The results obtained by the present investigation may be checked experimentally by preparing such two-level system and observing its time development by using probe em fields [1].

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